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ON THE SOLUTION OF CREEP THEORY PROBLEMS FOR AGEING BODIES WITH GROWING SLITS AND CAVITIES

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There is considered a problem of creep theory for ageing homogeneous linearly deformed bodies with growing slits and cavities. Only the stresses or displacements are given on the moving sections of the contour. It is assumed that the Poisson's ratio is constant. Explicit representations are obtained from the stresses, strains, and displacements of the creep theory problem in terms of the stresses, strains, and displacements of elastically instantaneous problems. In particular, it follows from these representations that for the problem under consideration the Volterra principle is invalid in the general case. The results obtained extend the known theorems of Arutiunian which are valid for a domain with fixed boundaries [1 —4]. The main results of the paper without the extension to the case of developing cavities were announced in [5].

1. Let a homogeneous isotropic linearly deformed body possessing the properties of ageing and creep occupy a three-dimensional domain $\Omega(\tau)$ within which the volume forces $\mathbf{f}(\mathbf{x}, \tau) = \{f_i(\mathbf{x}, \tau)\}$ and the forced strains $\varepsilon_{ij}^\circ(\mathbf{x}, \tau)$ are given. The Poisson's ratio ν is taken constant.

The boundary $S(\tau)$ of the domain $\Omega(\tau)$ consists of four fixed sections S_i ($i = 1, 2, 3, 4$), a quasistatically growing slit $\gamma(\tau)$ with the initial position $\gamma(t_0)$ and boundary $S_\omega(\tau)$ of a quasistatically growing cavity $\omega(\tau)$ with initial position $\omega(t_1)$. Here $t_0 \geqslant \tau_0$, $t_1 \geqslant \tau_0$, τ_0 is the time of the beginning of the application of the external effects, $\gamma(\tau_1) \subset \gamma(\tau_2)$, $\omega(\tau_1) \subset \omega(\tau_2)$ if $\tau_1 \leqslant \tau_2$. The stress vector $\mathbf{F}(\mathbf{x}, \tau) = \{F_i(\mathbf{x}, \tau)\}$, is given on S_1, the displacement vector $\mathbf{U}(\mathbf{x}, \tau) = \{U_i(\mathbf{x}, \tau)\}$ on S_2, the normal displacements $U_n(\mathbf{x}, \tau)$ and the tangential stress vector $\mathbf{F}_\tau(\mathbf{x}, \tau)$ on S_3, and the normal stresses $F_n(\mathbf{x}, \tau)$ and the tangential displacement vector $\mathbf{U}_\tau(\mathbf{x}, \tau)$ on S_4.

The Cauchy equations, equilibrium equations, and boundary conditions written for the time τ have the form

$$
\begin{aligned}
&\varepsilon_{ij}(\mathbf{x}, \tau) = 2^{-1}(u_{i,j}(\mathbf{x}, \tau) + u_{j,i}(\mathbf{x}, \tau)), \quad \sigma_{ij,j}(\mathbf{x}, \tau) + f_i(\mathbf{x}, \tau) = 0 \qquad (1.1)\\
&\mathbf{x} \in \Omega(\tau)\\
&\sigma_n(\mathbf{x}, \tau) = \mathbf{F}(\mathbf{x}, \tau), \quad \mathbf{x} \in S_1; \quad \mathbf{u}(\mathbf{x}, \tau) = \mathbf{U}(\mathbf{x}, \tau), \quad \mathbf{x} \in S_2\\
&u_n(\mathbf{x}, \tau) = U_n(\mathbf{x}, \tau), \quad \sigma_\tau(\mathbf{x}, \tau) = \mathbf{F}_\tau(\mathbf{x}, \tau), \quad \mathbf{x} \in S_3\\
&\sigma_n(\mathbf{x}, \tau) = F_n(\mathbf{x}, \tau), \quad \mathbf{u}_\tau(\mathbf{x}, \tau) = \mathbf{U}_\tau(\mathbf{x}, \tau), \quad \mathbf{x} \in S_4
\end{aligned}
$$

here ε_{ij}, σ_{ij} are the strain and stress tensors, $\mathbf{u} = \{u_i\}$ is the displacement field, $\mathbf{n} = \{n_i\}$ is the external normal to the boundary S, $\sigma_n = \{\sigma_{ij}n_j\}$ is the stress vector on the area with normal to $\mathbf{n}$; u_n, σ_n are the normal components of the displacement and stress vectors, $\mathbf{u}_\tau$, $\boldsymbol{\sigma}_\tau$ are the tangential displacement and stress vectors. The dependence between the strain and stress tensors has the form

$$\begin{gathered}
\varepsilon_{ij}(\mathbf{x},\tau) = (1+\nu)\left[(I+L)\left(\frac{\sigma_{ij}}{E}\right)\right] - \\
\nu\delta_{ij}\left[(I+L)\left(\frac{\sigma_{kk}}{E}\right)\right] + \varepsilon_{ij}^\circ(\mathbf{x},\tau), \quad \mathbf{x}\in\Omega(\tau) \\
\sigma_{ij}(\mathbf{x},\tau) = \frac{E(\tau)}{1+\nu}\left\{[(I+N)(\varepsilon_{ij}-\varepsilon_{ij}^\circ)] + \right. \\
\left. \delta_{ij}\frac{\nu}{1-2\nu}[(I+N)(\varepsilon_{kk}-\overset{\circ}{\varepsilon}_{kk})]\right\}, \quad \mathbf{x}\in\Omega(\tau) \\
I\varphi = \varphi(\mathbf{x},\tau), \quad L\varphi = \int_{\tau_0}^{\tau}\varphi(\mathbf{x},\xi)P(\tau,\xi)\,d\xi, \\
N\varphi = \int_{\tau_0}^{\tau}\varphi(\mathbf{x},\xi)R(\tau,\xi)\,d\xi
\end{gathered} \tag{1.2}$$

Here $E(\mathbf{x},\tau)\equiv E(\tau)$ is the modulus of the elastically instantaneous strains, $P(\tau,\xi)$ is the creep kernel, $R(\tau,\xi)$ is the relaxation kernel. The kernels $P(\tau,\xi)$ and $R(\tau,\xi)$ are interrelated by the dependence

$$R(t,\tau) + P(t,\tau) = -\int_{\tau}^{t} P(t,\xi)R(\xi,\tau)\,d\xi \tag{1.3}$$

Here a version is examined in which the stresses are given on the edges $\gamma^\pm(\tau)$ of the slit $\gamma(\tau)$ and on the boundary $S_\omega(\tau)$ of the cavity $\omega(\tau)$. In this case, (1.1) and (1.2) are supplemented by the boundary conditions

$$\begin{gathered}
\boldsymbol{\sigma}_n(\mathbf{x},\tau) = \mathbf{F}^\pm(\mathbf{x},\tau), \quad \mathbf{x}\in\gamma^\pm(\tau) \\
\boldsymbol{\sigma}_n(\mathbf{x},\tau) = \mathbf{F}(\mathbf{x},\tau), \quad \mathbf{x}\in S_\omega(\tau)
\end{gathered} \tag{1.4}$$

Let $u_i^*(\mathbf{x},\tau)$, $\varepsilon_{ij}^*(\mathbf{x},\tau)$, $\sigma_{ij}^*(\mathbf{x},\tau)$ denote the solution of the creep problem (1.1), (1.2), (1.4). Let us designate the problem (1.1), (1.2), (1.4) for zero deformative effects as Problem 1*, i.e., for $\varepsilon_{ij}^\circ = 0$, $\mathbf{U}=0$ on S_2, $U_n = 0$ on S_3, $\mathbf{U}_\tau = 0$ on S_4, and problem (1.1), (1.2), (1.4) for zero external forces as Problem 2*, i.e., for $f_i = 0$ in Ω, $\mathbf{F}=0$ on S_1, $\mathbf{F}_\tau = 0$ on S_3, $F_n = 0$ on S_4, $\mathbf{F}^\pm = 0$ on $\gamma^\pm$, $\mathbf{F}=0$ on S_ω^1. We let $u_i^{*(1)}$, $\varepsilon_{ij}^{*(1)}$, $\sigma_{ij}^{*(1)}$ and $u_i^{*(2)}$, $\varepsilon_{ij}^{*(2)}$, $\sigma_{ij}^{*(2)}$, respectively, denote the solutions of the problem 1*, 2*. Evidently

$$u_i^* = u_i^{*(1)} + u_i^{*(2)}, \quad \varepsilon_{ij}^* = \varepsilon_{ij}^{*(1)} + \varepsilon_{ij}^{*(2)}, \quad \sigma_{ij}^* = \sigma_{ij}^{*(1)} + \sigma_{ij}^{*(2)} \tag{1.5}$$

The purpose of this paper is to express the solutions of Problems 1*, 2* in terms of the solutions of the elastically instantaneous problems in which $\varepsilon_{ij}(\mathbf{x},\tau)$ and $\sigma_{ij}(\mathbf{x},\tau)$ are interrelated by the law of instantaneous elasticity

$$\varepsilon_{ij}(\mathbf{x},\tau) = (1+\nu)\frac{\sigma_{ij}(\mathbf{x},\tau)}{E(\tau)} - \nu\delta_{ij}\frac{\sigma_{kk}(\mathbf{x},\tau)}{E(\tau)} + \varepsilon_{ij}^\circ(\mathbf{x},\tau) \tag{1.6}$$

Let $u_i^{(1)}(\mathbf{x}, \tau, \gamma(\tau), \omega(\tau))$, $\varepsilon_{ij}^{(1)}(\mathbf{x}, \tau, \gamma(\tau), \omega(\tau))$, $\sigma_{ij}^{(1)}(\mathbf{x}, \tau, \gamma(\tau), \omega(\tau))$ denote the solution of the elastically instantaneous Problem 1 corresponding to the Problem 1*. It should satisfy Eqs. (1.1) and (1.6) for zero deformative effects, i.e., for $\varepsilon_{ij}^{\circ} = 0$ in $\Omega, \mathbf{U} = 0$ on S_2, $U_n = 0$ on S_3, $\mathbf{U}_\tau = 0$ on S_4, and conditions (1.4) on the moving part of the boundary.

Let $u_i^{(2)}(\mathbf{x}, \tau, \gamma(t), \omega(t))$, $\varepsilon_{ij}^{(2)}(\mathbf{x}, \tau, \gamma(t), \omega(t))$, $\sigma_{ij}^{(2)}(\mathbf{x}, \tau, \gamma(t), \omega(t))$ denote the solution of the elastically instaneous Problem 2 corresponding to the Problem 2*. It should satisfy Eqs. (1.1) and (1.6) for zero external forces, i.e., for $f_i = 0$ in Ω, $\mathbf{F} = 0$ on S_1, $\mathbf{F}_\tau = 0$ on S_3, $F_n = 0$ on S_4 and the following conditions on the moving sections of the boundary

$$\sigma_n(\mathbf{x}, \tau) = 0, \quad \mathbf{x} \in \gamma^{\pm}(t), \quad \mathbf{x} \in S_\omega(t) \tag{1.7}$$

Theorem 1. The solution of Problem 1* is representable in the form

$$\sigma_{ij}^{*(1)}(\mathbf{x}, t) = \sigma_{ij}^{(1)}(\mathbf{x}, t, \gamma(t), \omega(t)) \tag{1.8}$$

$$\varepsilon_{ij}^{*(1)}(\mathbf{x}, t) = \varepsilon_{ij}^{(1)}(\mathbf{x}, t, \gamma(t), \omega(t)) + \int_{\tau_0}^{t} \varepsilon_{ij}^{(1)}(\mathbf{x}, \tau, \gamma(\tau), \omega(\tau)) P(t, \tau)\, d\tau \tag{1.9}$$

$$u_i^{*(1)}(\mathbf{x}, t) = u_i^{(1)}(\mathbf{x}, t, \gamma(t), \omega(t)) + \int_{\tau_0}^{t} u_i^{(1)}(\mathbf{x}, \tau, \gamma(\tau), \omega(\tau)) P(t, \tau)\, d\tau \tag{1.10}$$

Theorem 2. The solution of Problem 2* is representable in the form

$$\sigma_{ij}^{*(2)}(\mathbf{x}, t) = \sigma_{ij}^{(2)}(\mathbf{x}, t, \gamma(t), \omega(t)) + E(t) \int_{\tau_0}^{t} \frac{\sigma_{ij}^{(2)}(\mathbf{x}, \tau, \gamma(t), \omega(t))}{E(\tau)} R(t, \tau)\, d\tau \tag{1.11}$$

$$\begin{aligned} \varepsilon_{ij}^{*(2)}(\mathbf{x}, t) = {} & \varepsilon_{ij}^{(2)}(\mathbf{x}, t, \gamma(t), \omega(t)) + \int_{\tau_0}^{t} \varepsilon_{ij}^{(2)}(\mathbf{x}, \tau, \gamma(t), \omega(t)) R(t, \tau)\, d\tau + \\ & \int_{\tau_0}^{t} \varepsilon_{ij}^{(2)}(\mathbf{x}, \tau, \gamma(\tau), \omega(\tau)) P(t, \tau)\, d\tau + \\ & \int_{\tau_0}^{t} P(t, \tau) \left\{ \int_{\tau_0}^{\tau} \varepsilon_{ij}^{(2)}(\mathbf{x}, \xi, \gamma(\tau), \omega(\tau)) R(\tau, \xi)\, d\xi \right\} d\tau \end{aligned} \tag{1.12}$$

$$\begin{aligned} u_i^{*(2)}(\mathbf{x}, t) = {} & u_i^{(2)}(\mathbf{x}, t, \gamma(t), \omega(t)) + \\ & \int_{\tau_0}^{t} u_i^{(2)}(\mathbf{x}, \tau, \gamma(t), w(t)) R(t, \tau)\, d\tau + \\ & \int_{\tau_0}^{t} u_i^{(2)}(\mathbf{x}, \tau, \gamma(\tau), \omega(\tau)) P(t, \tau)\, d\tau + \\ & \int_{\tau_0}^{t} P(t, \tau) \left\{ \int_{\tau_0}^{\tau} u_i^{(2)}(\mathbf{x}, \xi, \gamma(\tau)\, \omega(\tau)) R(\tau, \xi)\, d\xi \right\} d\tau \end{aligned} \tag{1.13}$$

P r o o f o f T h e o r e m 1. Let us confirm that the functions $\sigma_{ij}^{*(1)}$, $\varepsilon_{ij}^{*(1)}$, $u_i^{*(1)}$ satisfy Eqs. (1.1), (1.2), (1.4) of the creep problem in the case of zero deformative effects.

By the definition of the elastically instantaneous solution and by virtue of (1.8), the stresses $\sigma_{ij}^{*(1)}$ satisfy the equilibrium equations and the stress boundary conditions.

For any t the strains $\varepsilon_{ij}^{*(1)}$ are evidently related to $u_i^{*(1)}$ by the Cauchy relationships $\varepsilon_{ij}^{*(1)} = {}^1/_2 (u_{i,j}^{*(1)} + u_{j,i}^{*(1)})$.

Since $u_i^{(1)}$ satisfy the homogeneous boundary conditions on S_2 , S_3, and S_4, it follows from (1.10) that the functions $u_i^{*(1)}$ satisfy these same conditions.

Let us verify the creep law (1.2). Let us substitute the expression for $\varepsilon_{ij}^{(1)}$ from (1.6) into (1.9); taking into account that $\varepsilon_{ij}^{\circ} = 0$, we obtain by taking (1.8) into account

$$\varepsilon_{ij}^{*(1)}(\mathbf{x}, \tau) = (1+\nu)(I+L)\left(\frac{\sigma_{ij}^{*(1)}}{E}\right) - \nu\delta_{ij}(I+L)\left(\frac{\sigma_{kk}^{*(1)}}{E}\right)$$

This agrees with the first equation in (1.2), Q. E. D.

P r o o f o f T h e o r e m 2. Let us verify that the functions $\sigma_{ij}^{*(2)}$, $\varepsilon_{ij}^{*(2)}$, $u_i^{*(2)}$ satisfy equations (1.1), (1.2), (1.4) of the creep problem in the case of zero force effects.

We substitute $\sigma_{ij}^{*(2)}(\mathbf{x}, t)$ for arbitrary t and arbitrary $\mathbf{x} \in \Omega(t)$ into the equilibrium equations; we obtain

$$\sigma_{ij,j}^{*(2)}(\mathbf{x}, t) = \sigma_{ij,j}^{(2)}(\mathbf{x}, t, \gamma(t), \omega(t)) + E(t)\int_{\tau_0}^{t} \frac{\sigma_{ij,j}^{(2)}(\mathbf{x}, \tau, \gamma(t), \omega(t))}{E(\tau)} R(t, \tau)\, d\tau \qquad (1.14)$$

The right side in (1.14) is zero on the basis of the definition of the elastically instantaneous solution.

Let us confirm compliance with the stress boundary conditions. On the moving contour $\gamma^{\pm}(t)$ and the moving boundary $S_\omega(t)$

$$\begin{gathered} \sigma_{ij}^{*(2)}(\mathbf{x}, t)\, n_j = \sigma_{ij}^{(2)}(\mathbf{x}, t, \gamma(t), \omega(t))\, n_j + \\ E(t)\int_{\tau_0}^{t} \frac{\sigma_{ij}^{(2)}(\mathbf{x}, \tau, \gamma(t), \omega(t))\, n_j}{E(\tau)} R(t, \tau)\, d\tau \\ \mathbf{x} \in \gamma^{\pm}(t), \quad \mathbf{x} \in S_\omega(t) \end{gathered} \qquad (1.15)$$

The elastically instantaneous solutions satisfy the conditions $\sigma_{ij}^{(2)}(\mathbf{x}, \tau, \gamma(t), \omega(t))n_j = 0$, $\mathbf{x} \in \gamma^{\pm}(t)$, $\mathbf{x} \in S_\omega(t)$, $\tau \leqslant t$, hence, the right side in (1.15) is zero. Therefore, $\sigma_{ij}^{*(2)}(\mathbf{x}, t)n_j = 0$, $\mathbf{x} \in \gamma^{\pm}(t)$, $\mathbf{x} \in S_\omega(t)$. The stress boundary conditions on S_1, S_3, S_4 are verified analogously.

Let us confirm that the displacements $u_i^{*(2)}$ satisfy the boundary conditions on part of the boundary S_2 (the verfication of the displacement boundary conditions on S_3 , S_4 is performed analogously).

Since $u_i^{(2)}(\mathbf{x}, \tau, \gamma(t), \omega(t)) = U_i(\mathbf{x}, \tau)$ for $\mathbf{x} \in S_2$, we obtain from (1.13) that for $\mathbf{x} \in S_2$

$$u_i^{*(2)}(\mathbf{x},t) = U_i(\mathbf{x},t) + \int_{\tau_0}^{t} U_i(\mathbf{x},\tau) R(t,\tau) d\tau + \int_{\tau_0}^{t} U_i(\mathbf{x},\tau) P(t,\tau) d\tau + \int_{\tau_0}^{t} P(t,\tau) \left\{ \int_{\tau_0}^{\tau} U_i(\mathbf{x},\xi) R(\tau,\xi) d\xi \right\} d\tau = U_i(\mathbf{x},t) \tag{1.16}$$

This last equality follows from the identity

$$\int_{\tau_0}^{t} R(t,\tau)\varphi(\tau) d\tau + \int_{\tau_0}^{t} P(t,\tau)\varphi(\tau) d\tau + \int_{\tau_0}^{t} P(t,\tau) \left\{ \int_{\tau_0}^{\tau} R(\tau,\xi)\varphi(\xi) d\xi \right\} d\tau = 0 \tag{1.17}$$

which results from (1.3) and is valid for any function $\varphi(\tau)$ given for $\tau_0 \leqslant \tau \leqslant t$, whereupon the sum of the integral terms in (1.16) is zero.

The Cauchy equations are satisfied because $\varepsilon_{ij}^{*(2)}$ and $u_i^{*(2)}$ are expressed in terms of $\varepsilon_{ij}^{(2)}$ and $u_i^{(2)}$, respectively, by means of the same integral operator.

Let us confirm the creep law, i.e., we show that

$$\varepsilon_{ij}^{*(2)}(\mathbf{x},\tau) = (1+\nu)(I+L)\left(\frac{\sigma_{ij}^{*(2)}}{E}\right) - \nu\delta_{ij}(I+L)\left(\frac{\sigma_{kk}^{*(2)}}{E}\right) + \varepsilon_{ij}^{\circ}(\mathbf{x},\tau) \tag{1.18}$$

We invert the instantaneous elasticity law (1.6) in Problem 2

$$\sigma_{ij}^{(2)}(\mathbf{x},\tau,\gamma(t),\omega(t)) = \frac{E(\tau)}{1+\nu}\Big\{[\varepsilon_{ij}^{(2)}(\mathbf{x},\tau,\gamma(t),\omega(t)) - \varepsilon_{ij}^{\circ}(\mathbf{x},\tau)] + \delta_{ij}\frac{\nu}{1-2\nu}[\varepsilon_{kk}^{(2)}(\mathbf{x},\tau,\gamma(t),\omega(t)) - \overset{\circ}{\varepsilon}_{kk}(\mathbf{x},\tau)]\Big\} \tag{1.19}$$

Substituting (1.19) into (1.11), we obtain

$$\begin{aligned} \frac{\sigma_{ij}^{*(2)}(\mathbf{x},t)}{E(t)} = {} & \frac{1}{1+\nu}\Big\{[\varepsilon_{ij}^{(2)}(\mathbf{x},t,\gamma(t),\omega(t)) - \varepsilon_{ij}^{\circ}(\mathbf{x},t)] + \\ & \int_{\tau_0}^{t}[\varepsilon_{ij}^{(2)}(\mathbf{x},\tau,\gamma(\tau),\omega(\tau)) - \varepsilon_{ij}^{\circ}(\mathbf{x},\tau)] R(t,\tau) d\tau\Big\} + \\ & \delta_{ij}\frac{\nu}{(1+\nu)(1-2\nu)}\Big\{[\varepsilon_{kk}^{(2)}(\mathbf{x},t,\gamma(t),\omega(t)) - \overset{\circ}{\varepsilon}_{kk}(\mathbf{x},t)] + \\ & \int_{\tau_0}^{t}[\varepsilon_{kk}^{(2)}(\mathbf{x},\tau,\gamma(\tau),\omega(\tau)) - \overset{\circ}{\varepsilon}_{kk}(\mathbf{x},\tau)] R(t,\tau) d\tau\Big\} + \\ & \frac{1}{1+\nu}\int_{\tau_0}^{t}\varepsilon_{ij}^{(2)}(\mathbf{x},\tau,\gamma(t),\omega(t)) R(t,\tau) d\tau + \\ & \delta_{ij}\frac{\nu}{(1+\nu)(1-2\nu)}\int_{\tau_0}^{t}\varepsilon_{kk}^{(2)}(\mathbf{x},\tau,\gamma(t),\omega(t)) R(t,\tau) d\tau - \end{aligned} \tag{1.20}$$

$$\frac{1}{1+\nu}\int_{\tau_0}^{t} \varepsilon_{ij}^{(2)}(\mathbf{x}, \tau, \gamma(\tau), \omega(\tau)) R(t, \tau)\, d\tau -$$

$$\delta_{ij}\frac{\nu}{(1+\nu)(1-2\nu)}\int_{\tau_0}^{t} \varepsilon_{kk}^{(2)}(\mathbf{x}, \tau, \gamma(\tau), \omega(\tau)) R(t, \tau)\, d\tau$$

If the law (1.18) holds, then substituting (1.20) into (1.18) (τ must first be replaced by t and ξ by τ in (1.18)) should yield (1.12). Since

$$K(M(\varphi)) = M(K(\varphi)) = \varphi$$
$$K\varphi \equiv (I+L)\varphi, \quad M\varphi \equiv (I+N)\varphi$$

then substituting the first two terms in the right side of (1.20) (the expressions in the braces) into (1.18) yields the expression

$$\varepsilon_{ij}^{(2)}(\mathbf{x}, t, \gamma(t), \omega(t)) - \varepsilon_{ij}{}^{\circ}(\mathbf{x}, t)$$

Taking account of (1.19) and (1.6), the third and fourth terms in (1.20) yield terms of the form

$$\int_{\tau_0}^{t} \varepsilon_{ij}^{(2)}(\mathbf{x}, \tau, \gamma(t), \omega(t)) R(t, \tau)\, d\tau +$$
$$\int_{\tau_0}^{t} P(t, \tau)\left\{\int_{\tau_0}^{\tau} \varepsilon_{ij}^{(2)}(\mathbf{x}, \xi, \gamma(\tau), \omega(\tau)) R(\tau, \xi)\, d\xi\right\} d\tau$$

and the fifth and sixth terms yield

$$-\int_{\tau_0}^{t} \varepsilon_{ij}^{(2)}(\mathbf{x}, \tau, \gamma(\tau), \omega(\tau)) R(t, \tau)\, d\tau -$$
$$\int_{\tau_0}^{t} P(t, \tau)\left\{\int_{\tau_0}^{\tau} \varepsilon_{ij}^{(2)}(\mathbf{x}, \xi, \gamma(\xi), \omega(\xi)) R(\tau, \xi)\, d\xi\right\} d\tau =$$
$$\int_{\tau_0}^{t} \varepsilon_{ij}^{(2)}(\mathbf{x}, \tau, \gamma(\tau), \omega(\tau)) P(t, \tau)\, d\tau$$

The identity (1.17) was used in deriving the last expression.

Summing the expressions obtained together with the term $\varepsilon_{ij}{}^{\circ}(\mathbf{x}, t)$ yields the right side of (1.12), Q. E. D.

2. Now, let us examine the version in which the displacements are given on the edges of the growing slit and the boundary of the cavity $\omega(\tau)$. In this case, the conditions

$$\mathbf{u}(\mathbf{x}, \tau) = \mathbf{U}^{\pm}(\mathbf{x}, \tau), \quad \mathbf{x} \in \gamma^{\pm}(\tau) \tag{2.1}$$
$$\mathbf{u}(\mathbf{x}, \tau) = \mathbf{U}(\mathbf{x}, \tau), \quad \mathbf{x} \in S_{\omega}(\tau)$$

should be satisfied in addition to (1.1) and (1.2).

As before, we let u_i^*, ε_{ij}^*, σ_{ij}^* denote the solution of the creep problem.

We designate the Problem (1.1), (2.1), (1.2) with zero deformative effects the Problem 1*: $\varepsilon_{ij}^\circ = 0$ in Ω, $\mathbf{U} = 0$ in S_2.

$$U_n = 0 \quad \text{on} \quad S_3, \mathbf{U}_\tau = 0 \quad \text{on} \quad S_4, \mathbf{U}^\pm = 0 \quad \text{on} \quad \gamma^+(\tau), \mathbf{U} = 0 \quad \text{on} \quad S_\omega(\tau)$$

and we designate the Problem (1.1), (2.1), (1.2) for zero external forces as the Problem 2*. The solution of the elastically instantaneous Problem 1 corresponding to the Problem 1* will be denoted by $u_i^{(1)}(\mathbf{x}, \tau, \gamma(t), \omega(t))$, $\varepsilon_{ij}^{(1)}(\mathbf{x}, \tau, \gamma(t), \omega(t))$, $\sigma_{ij}^{(1)}(\mathbf{x}, \tau, \gamma(t), \omega(t))$. It should satisfy the Eqs. (1.1) and (1.6) for zero deformative effects and the following conditions on the slit and the cavity boundary

$$\mathbf{u}(\mathbf{x}, \tau) = 0, \quad \mathbf{x} \in \gamma^\pm(t); \quad \mathbf{u}(\mathbf{x}, \tau) = 0, \quad \mathbf{x} \in S_\omega(t)$$

The solution of the elastically instantaneous Problem 2 corresponding to the Problem 2* is denoted by $u_i^{(2)}(\mathbf{x}, \tau, \gamma(\tau), \omega(\tau))$, $\varepsilon_{ij}^{(2)}(\mathbf{x}, \tau, \gamma(\tau), \omega(\tau))$, $\sigma_{ij}^{(2)}(\mathbf{x}, \tau, \gamma(\tau), \omega(\tau))$. It should satisfy Eqs. (1.1), (1.6) and (2.1) for zero external forces.

As before, the representation (1.5) is evidently valid.

Theorem 3. The solution of the Problem 1* is representable as

$$\sigma_{ij}^{*(1)}(\mathbf{x}, t) = \sigma_{ij}^{(1)}(\mathbf{x}, t, \gamma(t), \omega(t)) + E(t)\int_{\tau_0}^{t} \frac{\sigma_{ij}^{(1)}(\mathbf{x}, \tau, \gamma(t), \omega(t))}{E(\tau)} \times \tag{2.2}$$

$$P(t, \tau)\, d\tau + E(t)\int_{\tau_0}^{t} \frac{\sigma_{ij}^{(1)}(\mathbf{x}, \tau, \gamma(\tau), \omega(\tau))}{E(\tau)} R(t, \tau)\, d\tau +$$

$$E(t)\int_{\tau_0}^{t} R(t, \tau)\left\{\int_{\tau_0}^{\tau} \frac{\sigma_{ij}^{(1)}(\mathbf{x}, \xi, \gamma(\tau), \omega(\tau))}{E(\xi)} P(\tau, \xi)\, d\xi\right\} d\tau$$

$$\varepsilon_{ij}^{*(1)}(\mathbf{x}, t) = \varepsilon_{ij}^{(1)}(\mathbf{x}, t, \gamma(t), \omega(t)) + \int_{\tau_0}^{t} \varepsilon_{ij}^{(1)}(\mathbf{x}, \tau, \gamma(t), \omega(t))\, P(t, \tau)\, d\tau \tag{2.3}$$

$$u_i^{*(1)}(\mathbf{x}, t) = u_i^{(1)}(\mathbf{x}, t, \gamma(t), \omega(t)) + \int_{\tau_0}^{t} u_i^{(1)}(\mathbf{x}, \tau, \gamma(t), \omega(t))\, P(t, \tau)\, d\tau \tag{2.4}$$

Theorem 4. The solution of the Problem 2* is representable as

$$\sigma_{ij}^{*(2)}(\mathbf{x}, t) = \sigma_{ij}^{(2)}(\mathbf{x}, t, \gamma(t), \omega(t)) + E(t)\int_{\tau_0}^{t} \frac{\sigma_{ij}^{(2)}(\mathbf{x}, \tau, \gamma(\tau), \omega(\tau))}{E(\tau)} R(t, \tau)\, d\tau \tag{2.5}$$

$$\varepsilon_{ij}^{*(2)}(\mathbf{x}, t) = \varepsilon_{ij}^{(2)}(\mathbf{x}, t, \gamma(t), \omega(t)), \quad u_i^{*(2)}(\mathbf{x}, t) = u_i^{(2)}(\mathbf{x}, t, \gamma(t), \omega(t)) \tag{2.6}$$

The proof of Theorems 3 and 4 is completely analogous to the proof of Theorems 2 and 1, and is not presented here.

R e m a r k s. 1°. In the case of fixed slits and cavities (i. e., for $\gamma(\tau) \equiv \gamma(t) \equiv \gamma(t_0)$, $\omega(\tau) \equiv \omega(t) \equiv \omega(t_1)$), the representations (1. 8) — (1. 10) and (2. 5), (2. 6) evidently agree with the known representations following from the first and second theorems of Arutiunian [1 — 4]. The representations (1. 11) — (1. 13) and (2. 2) — (2. 4) go over into the known representations following from the second and first theorems of Arutiunian [1 — 4]. This latter follows from the identity (1. 17) which cancels the integral terms in (1. 12), (1. 13) and (2. 2) for the case of fixed slits and cavities.

2°. For $E(t) \equiv E = \text{const}$, $P(t, \tau) \equiv P(t - \tau)$, $R(t, \tau) \equiv R(t - \tau)$ the results presented above go over into a dependence for a hereditary elastic of viscoelastic body with constant Poisson's ratio.

3°. Theorems 2 and 3 are the main result of the paper. The representations (1. 11) —(1. 13) and (2. 2) — (2. 4) show that in these cases the Volterra principle [6] is invalid. Indeed, the formal application of the Volterra principle to the conditions of Theorem 2 result in the dependences

$$\sigma_{ij}^{*(2)}(\mathbf{x}, t) = \sigma_{ij}^{(2)}(\mathbf{x}, t, \gamma(t), \omega(t)) + E(t) \int_{\tau_0}^{t} \frac{\sigma_{ij}^{(2)}(\mathbf{x}, \tau, \gamma(\tau), \omega(\tau))}{E(\tau)} R(t, \tau)\, d\tau$$

$$\varepsilon_{ij}^{*(2)}(\mathbf{x}, t) = \varepsilon_{ij}^{(2)}(\mathbf{x}, t, \gamma(t), \omega(t)), \quad u_i^{*(2)}(\mathbf{x}, t) = u_i^{(2)}(\mathbf{x}, t, \gamma(t), \omega(t))$$

which agrees with (1. 11) — (1. 13) only in the case of fixed slits and cavities. An analogous result is obtained for the conditions of Theorem 3. As regards the representations (1. 8) — (1. 10) and (2. 5), (2. 6) (Theorems 1 and 4), they can be obtained by a formal application of the Volterra principle. And, conversely, the proof of the representations (1. 8) — (1. 10) and (2. 5), (2. 6) can be considered as a proof of the applicability of the mentioned principle for the conditions of Theorems 1 and 4. Let us note that Theorems 1 and 4 have been formulated in [7] for a growing slit in an isotropic viscoelastic body (Theorem 1 under the additional assumption of simple-connectedness of the domain considered).

4°. Theorems 1 — 4 allow expression of the asymptotic of the solution of the creep problem in the neighborhood of a quasistatic moving slit in terms of the known asymptotics [8] of the solution of the elastic problem.

5°. On the basis of known existence and uniqueness solutions of elasticity theory problems [9 — 11], constructive representations of the solution of the creep problem in terms of the solution of the elasticity problems, obtained in the Theorems [1 — 4], permit proving the existence and uniqueness of the solution of the creep problem for bodies with developing slits and cavities.

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